Journal of Mathematics and Music, 2014
http://dx.doi.org/10.1080/17459737.2013.847978

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Constructing an optimal circulating temperament based on a set of musical requirements

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(*Received 28 June 2011; accepted 17 September 2013*)

A novel parametric circulating temperament is presented using a constructive approach. The temperament is optimal with respect to a heuristically chosen set of musical requirements. It is parametric in the sense that the tempering of the narrowest (i.e. closest to pure) major third can be freely chosen. Equal temperament arises as a limiting case. The temperament is optimal in the sense that the tempering of the widest major third and the narrowest fifth is minimized given the size of the least tempered major third. Also, under this constraint, the tempering of the major thirds closest to the least tempered third along the circle of fifths is minimized. The remaining degree of freedom is used to minimize the number of unique intervals. The resulting temperament exhibits various symmetries, and has, in general, two differently sized fifths and five differently sized major thirds. The temperament has no historic relevance as such, but can find good uses within all keyboard music from early baroque till today due to the selection of optimization criteria that closely follow historical requirements for good temperaments. With some exercise it can be tuned by ear.

Keywords: tuning; temperament; optimal; fifths; thirds

1. Introduction

Throughout the centuries, different cultures have been using different musical tuning systems. In western music, it is common practice to construct scales by selecting notes from 12-note gamuts. There are mainly two musical aspects to the selection of frequencies for the notes of the gamut; the melodic and the harmonic. It is well established that simple small integer ratios of frequencies, or close approximations of such, give the most consonant combinations of sound. Of particular importance between the pure intervals are the octave (2:1), fifth (3:2), major third (5:4), and combinations thereof. Upon the selection of the frequencies for a 12-note gamut, it is a common design criterion that these intervals are contained as relationships between the notes of the gamut. However, since many of these numbers are relatively prime, it is impossible to construct a gamut where all the contained intervals are pure in the sense that they are small integer ratios. Instead of having some pure and some unusable intervals, it has been common practice in western music since the renaissance to have several, or even all of the intervals slightly impure to different degrees. This is commonly referred to as tempering, and the result as a temperament. If the resulting temperament divides the octave into equal steps, it is called an equal temperament. Temperaments where all intervals are musically usable, but not necessarily to the same degree,

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are referred to as circulating temperaments. For a thorough introduction to the theory of tuning and temperaments, see, e.g. Benson (2007).

Many historical temperaments remain known to our time, and they have different properties in terms of to which degree they favour specific keys and tonalities, and in terms of the number of distinct intervals. An extensive overview of historical temperaments was provided by Lindley (1987). Recently, Duffin (2007) argued strongly against the common practice of using equal temperament for the performance of historical music. He also argued that the number of distinct intervals in a temperament should be minimized, and thus advocated the use of meantone temperaments (Duffin 2000).

Particular interest has been paid to the temperament allegedly intended by Johann Sebastian Bach in his "Das Wohltemperierte Clavier". It is by now fairly well agreed that he did not intend the use of equal temperament but instead a non-equal circulating temperament. Several hypothetical reconstructions of Bach's temperament have been made, including the ones by Kelletat (see Benson 2007), Kellner (1977), Barnes (1979), Lehman (2005a, 2005b), Jencka (2005; 2011), O'Donnell (2006). Contrary to Duffin, Lehman argued that a temperament should have many differently sized intervals in order to achieve key personalities. Amiot (2009) demonstrated that Lehman's temperament is superior to other temperaments known to be available at Bach's time with respect to a goodness measure based upon the Fourier transform of the resulting musical scales, but did not compare it to other suggested Bach temperaments.

Sethares (1994) invented a system with adaptive tuning, i.e. a tuning that adapts continuously and automatically to the combination of notes being played. Amongst keyboard instruments, however, such approaches can only be used for electronic or electronically controlled instruments. A measurement of goodness-of-fit which aims to be objective was developed by Hall (1973). The measure was based on the relative importance of keys and intervals, and was constructed as a weighted average of the tempering of these. Goldstein (1977) proposed a method to construct an optimal temperament. The goal of the method was to minimize the impurity of all fifths and major and minor thirds. He showed that several historical temperaments could be seen as optimal with respect to this criterion under different constraints. Sethares (1993) developed a consonance metric based upon the perceptual data of Plomp and Levelt (1965). Sankey and Sethares (1997) used this metric to construct an optimized temperament for the music of Domenico Scarlatti. Polansky et al. (2008) followed a similar path, but introduced the use of all intervals in all keys, and set individual priorities or weights to the keys and intervals. In this way, they were able to reconstruct historical temperaments such as Werckmeister III, Young's temperament (Benson 2007) and equal temperament quite closely. Recently, Milne et al. (2011) developed a similarity metric for pitch collections based on the novel concept of expectation tensors.

In the current paper, a different approach is followed. Instead of defining an objective metric of consonance as a starting point, a set of musical requirements is chosen. The criteria aim to follow historical temperaments, and can therefore be subject to debate. However, it is shown that given the choice of prioritized musical requirements, a parametric temperament that is optimal with respect to the selected requirements can be constructed. The following set of requirements is chosen and prioritized as follows: (1) there should be no wide fifths and no narrow major thirds, since wide fifths or narrow major thirds lead to unnecessarily strong tempering of other intervals. (2) There must be a tonal centre corresponding to the least tempered major third, and the major thirds closest to this tonal centre along the circle of fifths should have priority over the more distant ones. (3) The temperament should be as symmetric as possible about the tonal centre in terms of the major thirds. In other words, if C is chosen as the tonal centre, flat keys should not be favoured over sharps or vice versa. (4) The resulting number of unique intervals in the temperament should be kept as low as possible. (5) No interval must be tempered more than absolutely necessary in order to obey the other criteria.

2. Developing the temperament

2.1. *Notation and definitions*

Start by numbering the 12 notes along the circle of fifths (Figure 1), by $i \in \{0, \ldots, 11\}$, such that $C = 0$, $G = 1$, etc., using enharmonic equivalences, e.g. $D\# = E\flat = F\flat\flat$, etc. as commonly done for 12-note temperaments.

A temperament is completely described when, e.g. the sizes of the 12 fifths are known. Let $f(i)$ denote the tempering of the upward fifth starting at note i as measured on a logarithmic scale. In particular, $f(0)$ is the tempering of the fifth C–G. If $f(i) = 0$, the fifth from note i is pure. If $f(i) < 0$, the fifth is narrow, and if $f(i) > 0$, it is wide. In order to close the circle of fifths, the total tempering of the fifths must add up to the Pythagorean comma (Benson 2007)

$$\sum_{i=0}^{11} f(i) = -P. \tag{1}$$

For convenience, the notation $\tilde{f}$ is introduced for the periodic extension with period 12 of the function f. In other words $\tilde{f}(i) = f(i)$ for $i \in \{0 \ldots 11\}$ and $\tilde{f}(i) = \tilde{f}(i + 12n)$ for $n \in \mathbb{Z}$.

Let $t(i)$ denote the tempering of the upward major third starting at note i. The major third is made up of four consecutive fifths. If these fifths are all pure, the resulting interval is a Pythagorean major third, which is one syntonic comma, S, wide. Thus, using periodic extensions as above

$$\tilde{t}(i) = S + \sum_{j=i}^{i+3} \tilde{f}(j). \tag{2}$$

Similar equations can be constructed for the other intervals of the scale when needed.

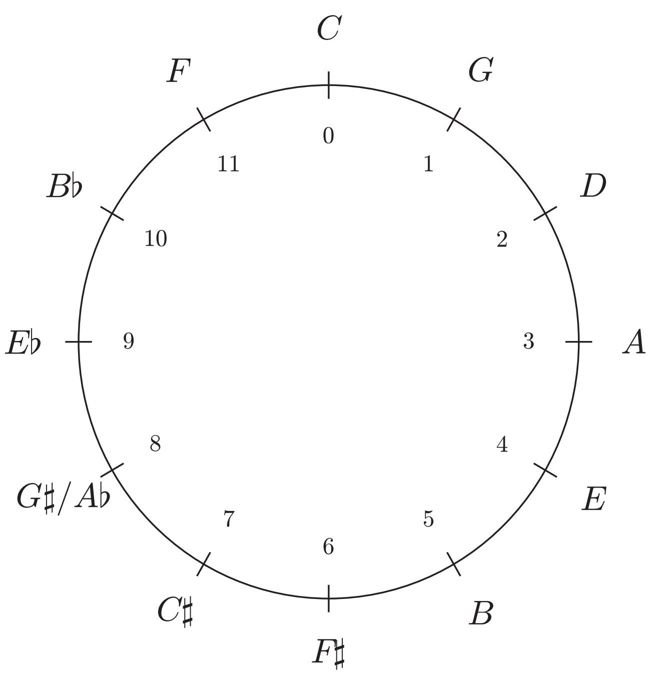

Figure 1. The circle of fifths. The numbers indicate the numbering system used throughout the text.

In a circulating temperament, three consecutive major thirds make up one octave. The difference between three pure major thirds and one octave is one diesis, which equals $3S - P$, as follows directly by combining Equations (1) and (2):

$$\sum_{j=0}^{2} \tilde{t}(i + 4j) = 3S - P, \quad \forall i. \tag{3}$$

2.2. *Basic assumptions*

In constructing the temperament, some assumptions must be made. In agreement with many (but not all) authors and historical temperaments (Benson 2007, Lindley 1987), it is here assumed that no major thirds are tuned narrow, and no fifths are tuned wide, i.e.

$$\tilde{f}(i) \leq 0, \quad \forall i, \tag{4}$$

$$\tilde{t}(i) \geq 0, \quad \forall i. \tag{5}$$

Non-equal temperaments most often have a main key of preference (Benson 2007). Without loss of generality, it can be assumed that this is the key of C. If a different tonal centre is desired, the temperament can easily be transposed. It is then preferred to have C–E as the least tempered major third, and thus

$$\tilde{t}(i) \geq \tilde{t}(0), \quad \forall i. \tag{6}$$

The less this major third is tempered, the less equal the resulting temperament becomes. In order to make a parametric temperament, the tempering of this major third is left to be specified by the user. No interval should be tempered more than necessary. This applies in particular to the fifths. Since $t(0)$ is the least tempered major third, the average tempering of the four fifths constituting it will have to be the most tempered sequence of four consecutive fifths (Figure 1). A minimized tempering of these fifths is obtained by distributing the total tempering $t(0)$ evenly across the first four fifths, giving

$$f(0) = f(1) = f(2) = f(3) = F_0, \tag{7}$$

where F_0 is the single parameter of choice. Thus, the tempering $t(0)$ is expressed as

$$t(0) = S + 4F_0, \tag{8}$$

hence F_0 must be chosen such that $F_0 \geq -S/4$ in order not to make $t(0) < 0$, which would disobey the criterion in Equation (5). Also, one must have $F_0 \leq -P/12$ according to Equations (1)–(3), else the assumption in Equation (6) of $t(0)$ be the least tempered major third will not hold true. Thus, one must have

$$-\frac{S}{4} \leq F_0 \leq -\frac{P}{12}. \tag{9}$$

Similarly, the tempering of the most tempered major thirds should be minimized given the tempering of the least tempered major third, Equation (8). According to Equation (3), a first step towards this goal can be achieved by setting

$$\tilde{t}(i) \leq t(4) = t(8), \quad \forall i. \tag{10}$$

A common property for historical temperaments is that the least tempered major thirds are close to each other along the circle of fifths (Figures 2 and 6). Since the tempering $t(4)$ and $t(8)$

is already given, no major third should be wider than these. However, in order to minimize the size of the major thirds close to the tonal centre, the major thirds between $t(4)$ and $t(8)$ should be as great as possible within the limit of Equation (10). Thus, one must have

$$t(4) = t(5) = t(6) = t(7) = t(8). \tag{11}$$

Solving Equation (11) for the fifths, using Equation (2), gives

$$\begin{aligned} f(4) &= f(8), \\ f(5) &= f(9), \\ f(6) &= f(10), \\ f(7) &= f(11). \end{aligned} \tag{12}$$

In total, Equations (1), (7) and (12) constitute 9 linear equations for the tempering of the 12 fifths, $f(i)$. This means that three degrees of freedom remain in addition to the designed-in freedom to select the parameter F_0. The remaining freedom can be used to set, e.g. three of the remaining major thirds $t(i)$, $i \in \{1, 2, 3, 9, 10, 11\}$, or three of the remaining fifths $f(i)$, $i \in \{4, 5, 6, 7, 8, 9, 10, 11\}$, any combination of three of these, or any other combination of three independent intervals not following from the nine equations already set.

Although this might seem quite some amount of freedom, the constraints formed by the nine equations following from the assumptions turn out to be quite strict. For example, very few of the existing temperaments known to the author obey all these constraints, the only ones being equal temperament (with $F_0 = -P/12$) and Johann Georg Neidhardt's circulating temperament no. 1 (with $F_0 = -P/6$). This latter temperament can be described by the vector whose components

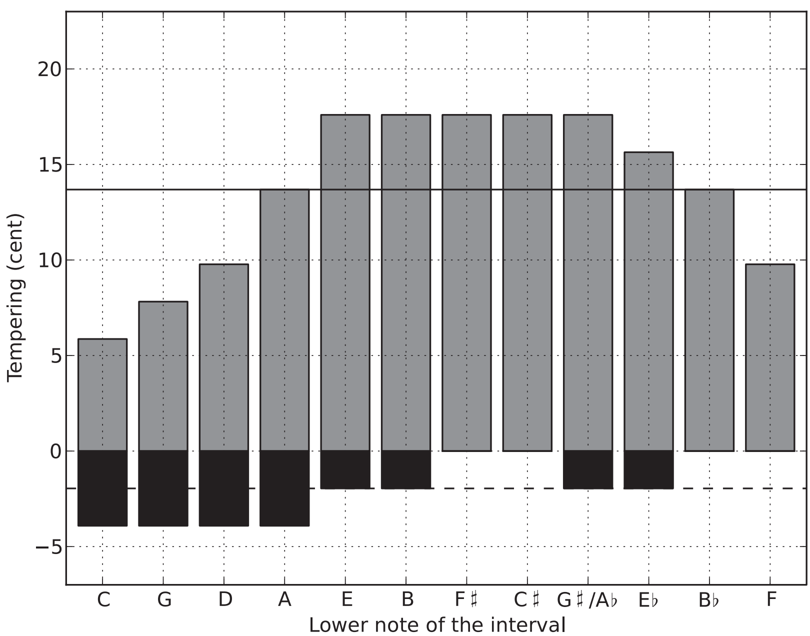

Figure 2. The tempering of fifths (black) and major thirds (grey) in Johann Georg Neidhardt's circulating temperament no. 1 as measured in cent. The solid and the dashed horizontal lines show the tempering of major thirds and fifths in equal temperament, respectively.

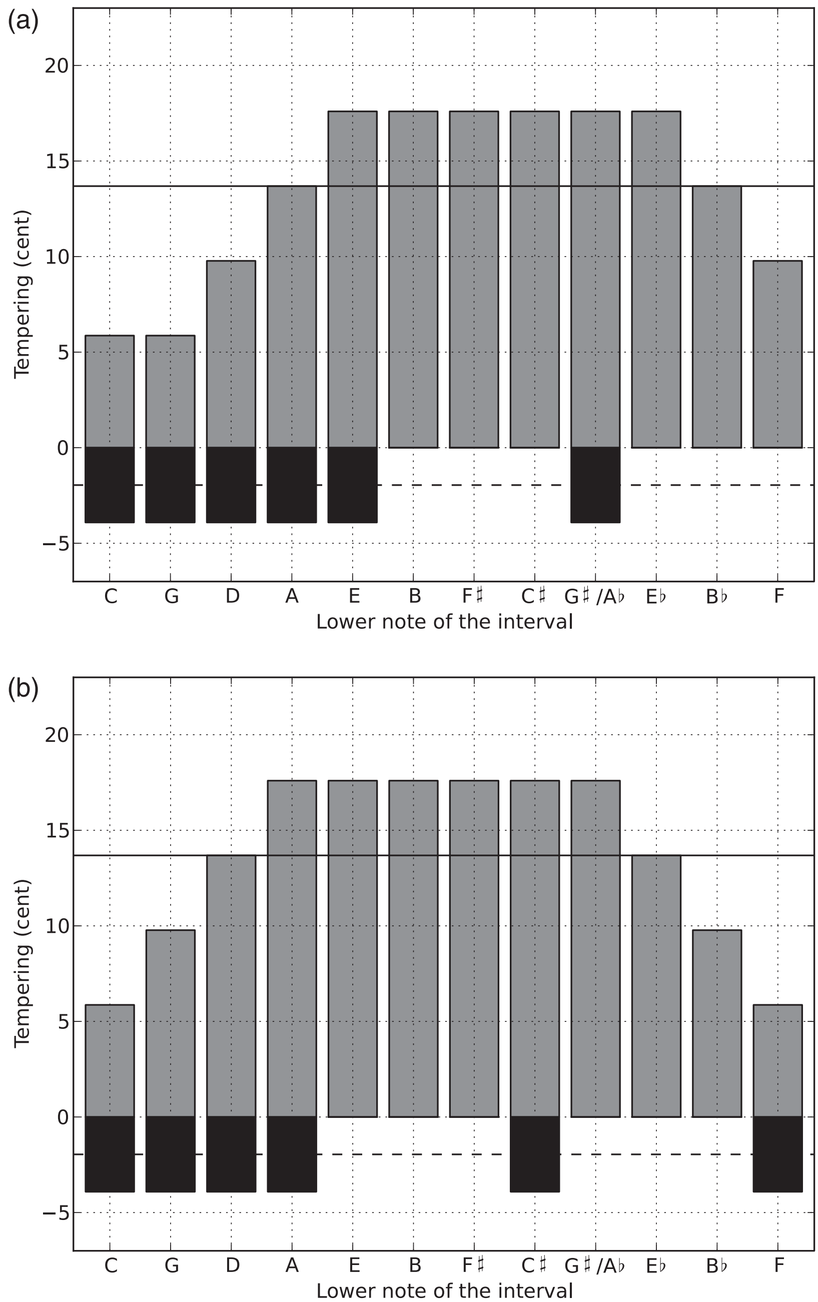

Figure 3. The tempering of fifths (black) and major thirds (grey) as measured in cent in constructed temperaments with priority for the sharp (above) and flat (below) keys. The solid and the dashed horizontal lines show the tempering of major thirds and fifths in equal temperament, respectively (see Section 2.2 for details).

are the tempering of the fifths $f(0), f(1), \ldots, f(11)$

$$\mathbf{f} = -\frac{P}{6}\begin{pmatrix}1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0\end{pmatrix}^{\mathrm{T}}. \tag{13}$$

An illustration of this temperament is shown in Figure 2. The figure shows the tempering of fifths and major thirds in cent. Cent is defined on a logarithmic scale such that one octave equals 1200 cent, and $P \approx 23.46001038$ cent.

The three remaining parameters can be used to favour keys with sharps or keys with flats as shown in Figure 3. These temperaments were obtained as extreme cases by setting $F_0 = -P/6$ and $f(9) = f(10) = f(11) = 0$ for the first one, giving

$$\mathbf{f} = -\frac{P}{6}(1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0)^{\mathrm{T}}, \tag{14}$$

and $f(4) = f(5) = f(6) = 0$ for the second one, giving

$$\mathbf{f} = \frac{P}{6}(1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1)^{\mathrm{T}}. \tag{15}$$

2.3. *Exploiting symmetries*

Although the possibility of constructing temperaments favouring flats or sharps can be interesting for particular applications, there seems to be no good reason for doing so for a general all-round temperament. Actually, for an optimal temperament, it is reasonable to insist on it being symmetric, i.e. that it does not favour flats over sharps or vice versa. This criterion can be formulated as the major thirds being symmetric about the tonal centre, which in this case is C

$$\tilde{t}(i) = \tilde{t}(-i), \quad \forall i. \tag{16}$$

Inserting the definitions of the major thirds, Equation (2) into Equation (16) and solving for $\tilde{f}(i)$ gives the equivalent symmetry of fifths

$$\tilde{f}(2+i) = \tilde{f}(2-i), \quad \forall i. \tag{17}$$

In other words, if the tempering of the major thirds is symmetric about the C–$F\sharp$ axis of the circle of fifths (Figure 1), the tempering of the fifths will be symmetric about the D–$A\flat$ axis. This result is independent of the assumptions made in Section 2.2, but follows directly from the definitions of the relationship between the tempering of the fifths and major thirds, Equation (2). Similar symmetry properties can be found for the other intervals in exactly the same way. For example, the major seconds, being made up of two consecutive fifths, will be symmetric about the G–$D\flat$ axis in the chosen case.

The symmetries in Equation (16), or, equivalently, Equation (17) constitute six linearly independent equations. Together with the nine equations from the basic assumptions in Section 2.2, there are now 15 equations for the 12 unknown fifths. However, they are not all linearly independent, as can be seen, e.g. by writing the equations in matrix form and calculating the rank of the system matrix. Only two of the symmetry equations are linearly independent from the nine equations already established. Thus, even with the symmetry criterion, there is still one remaining degree of freedom.

The general solution to the set of 11 Equations (1), (7), (12) and (16) can be written as

$$\mathbf{f} = \begin{pmatrix} F_0 \\ F_0 \\ F_0 \\ F_0 \\ F_1 \\ -F_0 - F_1 - \frac{P}{4} \\ -F_0 - F_1 - \frac{P}{4} \\ F_1 \\ F_1 \\ -F_0 - F_1 - \frac{P}{4} \\ -F_0 - F_1 - \frac{P}{4} \\ F_1 \end{pmatrix}. \quad (18)$$

The effect of changing F_0 should be familiar by now; it determines how close to pure the best major third is. The effect of changing F_1 can be studied by setting it to the extreme values of $F_1 = -F_0 - P/4$ and $F_1 = 0$. The resulting temperaments in this case for $F_0 = -P/6$ are shown in Figure 4. Only the four major thirds $t(1)$, $t(3)$, $t(9)$ and $t(11)$ are affected. Setting F_1 as low as possible, i.e. tempering the *E–B* fifth as much as possible within the given constraints, results in a tempering favouring the major thirds closest to the central key along the circle of fifths, whereas setting $F_1 = 0$ gives priority to the more distant major thirds at the cost of increasing the tempering of the close major thirds. With respect to the criterion of prioritizing the most central major thirds, this should mean that the optimal temperament given F_0 is found by setting $F_1 = -F_0 - P/4$. However, there is an even better way to use the remaining degree of freedom.

2.4. *Number of distinct intervals*

Duffin (2000) argued strongly that the number of distinct intervals in a circulating temperament should be minimized. This is particularly important when playing together with other instruments, such as bowed string instruments without fixed pitches. Actually, Duffin goes as far as to promote the use of $\frac{1}{6}$ -comma meantone temperament. As a general temperament for keyboards for a broad range of music, this is not an option due to the number of unusable (wolf) intervals, but the criterion of reducing the number of distinct intervals can be applied also in the current setting. It should be noted that this is completely opposite to Lehman's requirement that a good temperament should have as many distinct interval as possible in order to achieve key personalities (Lehman 2005a, 2005b). Here, the Duffin criterion is chosen. Whether this is a good choice or not is a matter of taste and practical considerations that should be left to the discretion of the performing musicians.

In a temperament, as soon as the sizes of all the fifths are known, the sizes of the other intervals can be computed. The tempering of the major seconds equals the sum of two consecutive fifths, major sixths are three consecutive fifths, major thirds are four, major sevenths are five and the tritones are six. The remaining intervals are inversions of these. In the general case, the temperament in Equation (18) has three different fifths (and fourths), five major seconds (and minor sevenths), five major sixths (and minor thirds), five major thirds (and minor sixths), five major sevenths (and minor seconds) and seven tritones, adding up to a total of 30 different intervals, not counting inversions. This can be reduced, and has a unique minimum, which is obtained by setting $F_1 = -F_0/2 - P/8$. In this case, the eight fifths $f(4), \ldots, f(11)$ become equal, since

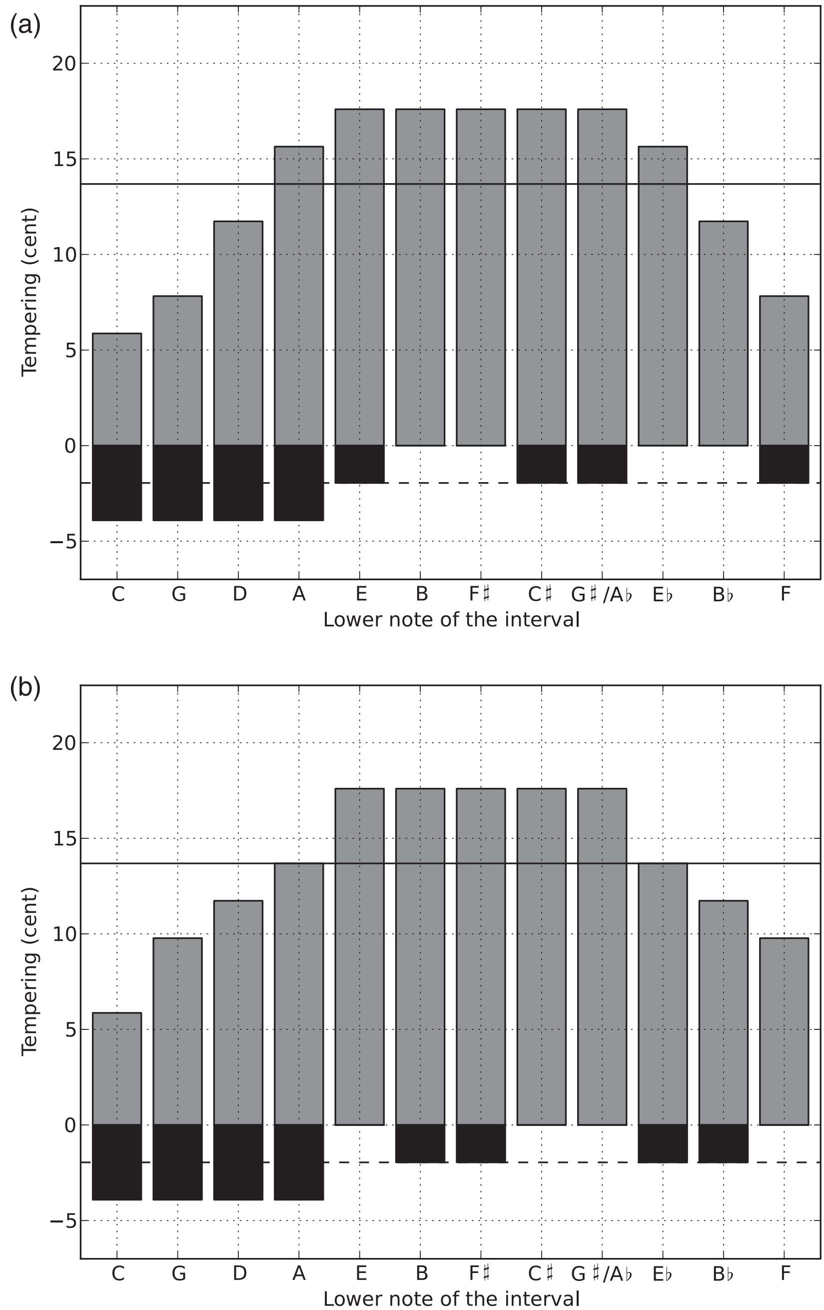

Figure 4. The tempering of fifths (black) and major thirds (grey) as measured in cent in constructed temperaments with priority for the central (above) and distant (below) keys. The solid and the dashed horizontal lines show the tempering of major thirds and fifths in equal temperament, respectively (see Section 2.3 for details).

$-F_0 - F_1 - P/4 = -F_0/2 - P/8$. The temperament is then completely described by

$$\mathbf{f} = \begin{pmatrix} F_0 \\ F_0 \\ F_0 \\ F_0 \\ -\frac{F_0}{2} - \frac{P}{8} \\ -\frac{F_0}{2} - \frac{P}{8} \\ -\frac{F_0}{2} - \frac{P}{8} \\ -\frac{F_0}{2} - \frac{P}{8} \\ -\frac{F_0}{2} - \frac{P}{8} \\ -\frac{F_0}{2} - \frac{P}{8} \\ -\frac{F_0}{2} - \frac{P}{8} \\ -\frac{F_0}{2} - \frac{P}{8} \end{pmatrix}. \tag{19}$$

This particular temperament has two distinct fifths, three major seconds, four major sixths, five major thirds, five major sevenths and five tritones, adding up to a total of 24 intervals, not counting inversions. The resulting temperament is shown in Figure 5 for various choices of F_0.

For some special choices of F_0, there are other even more optimal solutions with respect to this criterion. For $F_0 = -P/8$ there are two ways to achieve an even lower number of distinct intervals. Setting $F_1 = -P/8$ in Equation (18) gives a total of 20 different intervals, and setting $F_1 = 0$ gives a total of 17 different intervals. These are hence referred to as the suboptimal and optimal temperaments for $F_0 = -P/8$. For $F_0 = -P/12$, the resulting limiting case is equal temperament, where there is only one version of each interval, adding up to a total of six distinct non-unison intervals, not counting inversions.

3. Discussion

3.1. *Comparison with other temperaments*

Figure 6 shows the behaviour of well-known temperaments. As a general observation, it should be noted that the overall behaviour is not as systematic and symmetric as for the proposed solution in Figure 5. This might be interpreted as the proposed solution being superior to the historic temperaments in this respect, but it could also be taken as an observation undermining criterion (3) in Section 1.

Properties of well-known temperaments are shown together with the proposed optimal temperament for several choices of F_0 in Table 1. The temperaments are sorted by the size of the best (i.e. least tempered) major third, the size of the worst (i.e. most tempered) major third and the total number of distinct intervals not counting inversions, in order of priority. The table also shows the number of distinct version of the individual intervals. For each size of the best major third, the corresponding version of the suggested temperament is shown. For all of the temperaments listed, the proposed solution has the smallest size of the largest major third, and in many cases also the smallest number of distinct intervals.

Although not the major topic of the current paper, it is interesting to compare the hypothetical Bach temperaments in this respect. Two of the temperaments, Kelletat and Lehman, have a very high number of distinct intervals, whereas three others, Kellner, Barnes and O'Donnell, have a

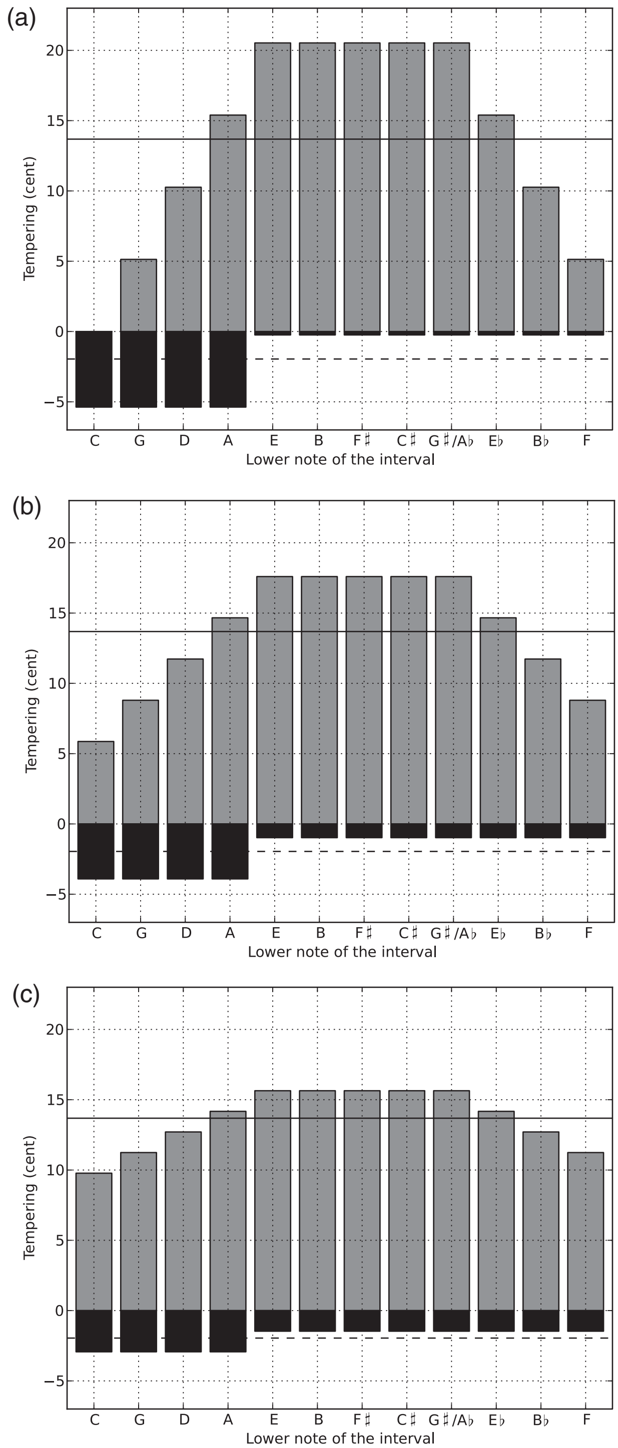

Figure 5. The tempering of fifths (black) and major thirds (grey) as measured in cent for the different choices of $F_0 = -S/4$ (top), $F_0 = -P/6$ (middle) and $F_0 = -P/8$ (bottom). The solid and the dashed horizontal lines show the tempering of major thirds and fifths in equal temperament, respectively (see Section 2.4 for details).

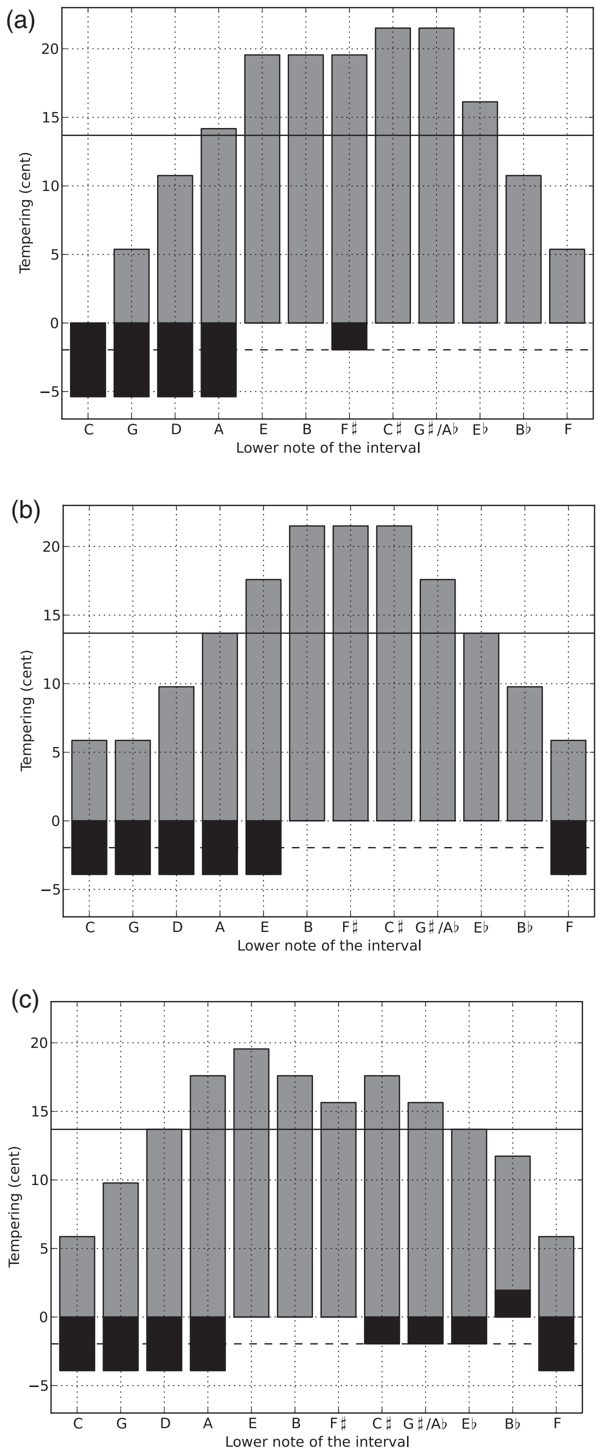

Figure 6. The tempering of fifths (black) and major thirds (grey) as measured in cent for well-known temperaments. Kirnberger III (top), Valotti (middle) and Lehman's Bach temperament (bottom). The solid and the dashed horizontal lines show the tempering of major thirds and fifths in equal temperament, respectively.

Table 1. Comparison of the suggested optimal temperament with well-known existing temperaments.

	Major thirds (tu)		Number of distinct intervals						
	Min	Max	5th	2nd	6th	3rd	7th	Tritones	Total
Pure	0.00	41.06	3	3	4	2	4	4	20
1/4-comma meantone	0.00	41.06	2	2	2	2	2	2	12
Kelletat	0.00	21.51	4	6	8	9	10	10	47
Kirnberger III	0.00	21.51	3	4	5	7	8	8	35
Proposed, $F_0 = -S/4$	0.00	20.53	2	3	4	5	5	5	24
Kellner	2.74	21.51	2	3	4	5	5	6	25
Proposed, $F_0 = -P/5$	2.74	19.16	2	3	4	5	5	5	24
Werckmeister III	3.91	21.51	2	3	4	4	4	5	22
Proposed, $F_0 = -3P/16$	3.91	18.57	2	3	4	5	5	5	24
Valotti	5.87	21.51	2	3	4	5	6	7	27
Lehman	5.87	19.55	4	4	6	7	8	9	38
Barnes	5.87	21.51	2	3	4	5	6	5	25
Neidhardt no. 1	5.87	17.60	3	5	5	6	7	9	35
Proposed, $F_0 = -P/6$	5.87	17.60	2	3	4	5	5	5	24
Sorge	9.78	17.60	3	5	5	4	6	7	30
Neidhardt 4	9.78	17.60	4	4	5	4	6	5	28
O'Donnell	9.78	17.60	3	4	5	4	5	5	26
Proposed, $F_0 = -P/8$	9.78	15.64	2	3	4	5	5	5	24
Suboptimal, $F_0 = -P/8$	9.78	15.64	2	3	3	3	4	5	20
Optimal, $F_0 = -P/8$	9.78	15.64	2	3	3	3	3	3	17
Equal temperament	13.69	13.69	1	1	1	1	1	1	6

Note: The temperaments are sorted by the best major third, the worst major third (in cent) and the total number of distinct intervals, in order of priority.

very low number. To the best of the author's knowledge, this particular aspect has not been much debated in the construction of Bach temperaments.

According to Table 1, the proposed temperament is superior to the other selected temperaments with respect to the chosen set of musical requirements. However, this does not necessarily mean that it is in any sense better than other temperaments. Appreciating the subtle nuances of different temperaments is something that has to be learned and trained, and in the end, people end up preferring different solutions. It is therefore the author's opinion that it does not make very much sense to perform perceptual experiments with the goal of showing that some temperament is better than some other. The only way to really judge a temperament (like any other subject of taste) is to try it out on a real acoustic instrument and make up one's own opinion. Thus, the following two paragraphs solely represent the author's personal opinion, and is not supposed to represent any scientific result.

With $F_0 = -S/4$, the resulting temperament in many ways resembles Kirnberger III. The narrowest major third is pure, and there are big differences in key personalities. However, the widest major third is less tempered than in Kirnberger III. Also, while Kirnberger III is quite asymmetric favouring keys with sharps, the proposed solution is much more symmetric. The major triads on E, B and $F\sharp$ are somewhat better in Kirnberger III, but all other triads sound better to the author in the proposed solution.

With $F_0 = -P/6$, the resulting temperament shares the size of the best major thirds with both the Valotti and Lehman temperaments. However, it differs from Valotti in that the worst major third is much better, thus giving an improved rendering of the keys distant from the tonal centre. It differs from the Lehman temperament in that the key personalities are not as strong. This may be judged as a drawback or as a benefit depending on taste. The keys with sharps are generally better with the proposed solution, whereas the keys with flats, $A\flat$ major, $E\flat$ major and F major in particular, sound more pleasing in Lehman's temperament. The E–$G\sharp$ major third is significantly purer in the proposed solution compared with Lehman's temperament. The number of distinct intervals is also very different between these two temperaments. Thus, the melodic lines might be less interesting, but more smooth with the proposed temperament, but, according to Duffin (2000) the proposed solution will be easier to adapt to for musicians playing bowed string instruments.

3.2. *Practical tuning recipe*

With some training, the temperament in Equation (19) can be tuned by ear on keyboard instruments. Here is a brief outline on how it can be achieved: temper the major third C–E as preferred; it should be pure or slightly wide. The choice prescribes the single parameter F_0. Then, tetrasect the major third into four equal fifths according to common procedure (for detailed instructions on how to perform the tetrasection of the major third by ear, see, e.g. Bavington 2007). Tune $G\sharp/A\flat$ such that the major thirds E–$G\sharp$ and $A\flat$–C are equally wide. Finally, tetrasect the major thirds E–$G\sharp$ and $A\flat$–C. With some exercise, this can be performed quite accurately and rapidly by ear.

4. Conclusion

A parametric circulating temperament is constructed. It is optimal with respect to a heuristically selected set of prioritized musical requirements, and, thus superior to other well-known temperaments with respect to the chosen criteria. The criteria are of course subject to debate, but if the criteria are agreed upon, the resulting temperament is shown to be optimal. According to the author's personal opinion, it lends itself well to a broad range of musical genres.

Acknowledgment

The author would like to thank the editors and the reviewers for fruitful comments and suggestions for the manuscript.

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