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Anisotropic Diffusion in Riemannian Colour Geometry

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Abstract

Anisotropic diffusion has long been an important tool in image processing. More recently, it has also found its way to colour imaging. Until now, mainly Euclidean colour spaces have been considered in this context, but recent years have seen a renewed interest in and importance of non-Euclidean colour geometry. The main contribution of this paper is the derivation of the equations for anisotropic diffusion in Riemannian colour geometry. It is demonstrated that it contains several well-known solutions such as Perona–Malik diffusion and Tschumperlé–Deriche diffusion as special cases. Furthermore, it is shown how it is non-trivially connected to Sochen's general framework for low-level vision. The main significance of the method is that it decouples the coordinates used for solving the diffusion equation from the ones that define the metric of the colour manifold, and thus directs the magnitude and direction of the diffusion through the diffusion tensor. It also enables the use of non-Euclidean colour manifolds and metrics for applications such as denoising, inpainting, and demosaicing, based on anisotropic diffusion.

Keywords Image processing · Colour geometry · Riemannian geometry · Anisotropic diffusion

1 Introduction

Since the mid-eighties, diffusion methods have been actively used for various purposes in image processing. Starting with simple linear diffusion for the denoising of greyscale images [1], it has since been extended to nonlinear diffusion [2], diffusion processes in colour images [3], and anisotropic diffusion of colour images [4] with applications to denoising and inpainting. More recently, the diffusion-based method of gradient domain image processing [5] has also found its way to colour imaging for such applications as HDR image tone mapping [6], image compression [7], gamut mapping [8], demosaicing [9], colour-to-greyscale conversion [10], and daltonisation [11].

Although it has long been established that perceptual colour manifolds are non-Euclidean [12–16], Euclidean colour spaces have been the primary consideration in the context of image diffusion until now. Recent years have shown a renewed interest in and importance of non-Euclidean colour geometries [17–20]. This emphasises the need of being able to perform basic image processing operations in such colour geometries.

In this paper, we derive the equations for anisotropic diffusion in Riemannian colour geometry as an extension of the methods of Perona and Malik [2] and Tschumperlé and Deriche [4]. Firstly, in Sect. 2, the state-of-the-art Riemannian colour geometry and diffusion methods in image processing and colour imaging are reviewed. The anisotropic diffusion equation in Riemannian geometry is then derived in detail in Sect. 3. In Sect. 4, it is demonstrated that it contains several well-known solutions as special cases.

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2 Background

Here, we review the state of the art for Riemannian colour geometry as well as for diffusion methods in image processing and colour imaging. To emphasise the similarities and differences between various approaches, we will use a consistent formulation and notation, thus deviating from that of the original sources. As such, it will also serve to introduce

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the notation used in the derivation of our method in the next section.

2.1 Riemannian Colour Geometry

Most literature on image processing for vector-valued images is performed in Euclidean colour geometries. In these cases, it makes the most sense to use a vectorial notation and not focus as much on the vector components. Shifting to a non-Euclidean geometry, the component-free notation becomes more problematic and will obscure many of the details in the mathematical derivations. We will therefore use a coordinate-dependent notation for Riemannian geometry [21, 22].

Throughout, we will use Latin indices for the spatial coordinates in the image domain and Greek indices for colour coordinates, whether in Euclidean or non-Euclidean geometries. Superscripts will denote the contravariant vector components, and subscripts will be used for the covariant ones. A comma before an index denotes partial differentiation with respect to that coordinate. The summation convention of Einstein [23] will be used throughout, unless otherwise specified. The convention says that every coordinate index that occurs twice in the same term implies a sum over that term, e.g. $a_i b_i = \sum_i a_i b_i$.

Already when Riemann [24] presented his theory of non-Euclidean geometry, mentioned colour perception as a potential application.[1] A central element in Riemannian geometry is the metric tensor g with components $g_{\mu\nu}$ that determines the line element ('infinitesimal distance') through the quadratic form

$$dl^2 = g_{\mu\nu} du^\mu du^\nu. \tag{1}$$

The first representation of a colour metric ('perceptual distance') described in terms of a Riemannian metric was given by Helmholtz [25]. Motivated by the Weber–Fechner law [26], it was described in terms of the line element

$$dl^2 = \left(\frac{du^0}{u^0}\right)^2 + \left(\frac{du^1}{u^1}\right)^2 + \left(\frac{du^2}{u^2}\right)^2 \tag{2}$$

corresponding to a diagonal metric tensor with $g_{\mu\mu} = 1/(u_\mu)^2$ on the diagonal (no sum). The colour coordinates correspond to the responses of the human visual system to long, medium, and short wavelengths, respectively. The resulting geometry is isometric with Euclidean geometry through the coordinate mapping $x^\mu = \ln u^\mu$.

The first actual non-Euclidean colour geometry was proposed by Schrödinger [12] (see Niall [27] for a modern translated and commented version) in terms of the metric,

$$dl^2 = \frac{1}{\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3} \times \left(\frac{\alpha_1\, du_1^2}{u_1} + \frac{\alpha_2\, du_2^2}{u_2} + \frac{\alpha_3\, du_3^2}{u_3}\right), \tag{3}$$

again corresponding to a diagonal metric with $g_{\mu\mu} = \alpha_\mu/(u^\mu \alpha_\nu u^\nu)$ (no sum for μ) on the diagonal. Stiles [13] went back to a form more similar to that of Helmholtz, but with relative weights to the three colour channels,

$$dl^2 = \left(\frac{du^0}{\alpha_0 u^0}\right)^2 + \left(\frac{du^1}{\alpha_1 u^1}\right)^2 + \left(\frac{du^2}{\alpha_2 u^2}\right)^2 \tag{4}$$

or $g_{\mu\mu} = (\alpha_\mu u^\mu)^{-2}$ (no sum). Motivated by research on colour vision, more elaborate extensions along this line were given by Bouman, Vos, and Walraven [14]. More recently, Pant and Farup [17] found that it is constructive to formulate industrial colour difference formulae within this framework. This applies even to non-Riemannian metrics such as CIEDE2000 [28] through the process of 'Riemannisation' [17].

Common to the non-Euclidean colour representations above is that they exhibit some form of negative curvature within iso-luminance surfaces. This is in complete agreement with the observations of Judd [15] on the phenomenon of 'hue super-importance', as well as with the results of MacAdam [29–31] on the curvature of the colour manifold based on observed colour discrimination. Based on a set of axioms essentially consisting of Grassmann's laws of colour vision augmented with the concept of homogeneity of the colour manifold (which is still open for debate), Resnikoff [16] proved that the geometry of colour perception must be either Euclidean or isomorphic to $\mathbb{R}^+ \times SL(2,\mathbb{R})/SO(2)$. The latter case can be realised as the Cartesian product $\mathbb{R}^+ \times \mathbb{H}$ of a positive real line (for brightness) and a hyperbolic plane (for chromaticity) in the Poincaré half-plane representation, resulting in the metric

$$dl^2 = \left(\frac{du^0}{u^0}\right)^2 + \alpha\left[\frac{(du^1)^2 + (du^2)^2}{(u^2)^2}\right] \tag{5}$$

again corresponding to a diagonal metric with $g_{00} = 1/(u^0)^2$, $g_{11} = g_{22} = \alpha/(u^2)^2$. It was shown by Farup [18] that enforcing this geometry (or rather the isometric Poincaré-disc representation, $\mathbb{R}^+ \times \mathbb{D}$) on already optimised Euclidean colour difference formulae consistently improved their performance. Recently, the tradition of Resnikoff has been revived and extended by Provenzi and co-workers, see [19, 20], clearly indicating that non-Euclidean colour geometry will become increasingly important in the future.

[1] '[...] dass die Orte der Sinnengegenstände und die Farben wohl die einzigen einfachen Begriffe sind, deren Bestimmungsweisen eine mehrfach ausgedehnte Mannigfaltigkeit bilden' [24]

2.2 Diffusion Methods for Grey-Scale Images

Diffusion processes can be found almost everywhere in nature. For general physical systems, they contribute to smoothing out quantities like temperature, heat, and concentration of chemical substances. The underlying mechanism is described by Fick's law of diffusion, saying that the flux of a diffusive quantity is directed opposite to the gradient of the same quantity. In other words, diffusive quantities tend to move from regions of high concentration to regions of low concentration. When Fick's law is combined with a conservation law for the same quantity, the diffusion (or heat) equation results

$$\frac{\partial u}{\partial t} = \nabla^2 u = u_{,ii}. \tag{6}$$

The Green's function of the diffusion equation is a Gaussian function with a variance that increases with time, showing that solving the diffusion equation is equivalent to performing the convolution with a Gaussian kernel.

The latter observation led to the introduction of diffusion methods for image processing. For image denoising of greyscale images, the technique was first used by Koenderink [1] by applying the heat equation directly to the pixel values. Koenderink also derived the same method from the criteria of causality, homogeneity, and isotropy.

While performing well for the smoothing of noisy areas, linear isotropic diffusion also blurs edges and details. To stop the diffusion at edges, Perona and Malik [2] introduced an image-dependent, local, nonlinear diffusion method described by the equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(s)\nabla u) = \partial_i (D(s) u_{,i}), \tag{7}$$

where $s = |\nabla u|^2$, describing the image 'structure', has been introduced. They proposed two different diffusion coefficients with different properties

$$D(s) = \exp\left(-\frac{s}{K^2}\right) \tag{8}$$

$$D(s) = \frac{1}{1 + s/K^2}. \tag{9}$$

Instead of designing PDEs directly, Rudin, Osher, and Fatemi [32] used a variational approach. The standard linear diffusion, Equation(6) is obtained by the Euler–Lagrange equations minimising the cost functional

$$E = \frac{1}{2}\int_\Omega |\nabla u|^2 d\Omega, \tag{10}$$

whereas they introduced the concept of total variation and showed that the minimisation of

$$E = \int_\Omega |\nabla u|\, d\Omega \tag{11}$$

leads to the PDE

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right). \tag{12}$$

Comparing this with the Perona–Malik diffusion, Equation (7), we see that it can be written in the same form choosing

$$D(s) = \frac{1}{\sqrt{s}}. \tag{13}$$

2.3 Data Attachment for Grey-Scale Images

The diffusion processes described above will produce a homogeneous grey image as $t \to \infty$ when solved with natural boundary conditions. One way to obtain the desired degree of image regularisation is to stop after a certain number of time steps. Alternatively, one can introduce a data attachment term

$$E = \frac{\omega}{2}\int_\Omega (u - u_0)^2 d\Omega \tag{14}$$

(also called 'fidelity term') to the cost functional to be minimised. This leads to an extra term $\omega(u - u_0)$ in the PDE. E.g. Equation (7) becomes:

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(s)\nabla u) - \omega(u - u_0). \tag{15}$$

2.4 Diffusion Methods for Colour Images

For the simple linear diffusion described by Equation (6), the extension to colour images is straightforward in the Euclidean case. Starting from the variational principle Equation (10), the colour channels of the image decouple, and we end up with identical linear diffusion equations for each of the colour channels.

For total variation and Perona–Malik diffusion, the extension is not that straightforward. Blomgren and Chan [3] introduced the concept of colour total variation by introducing a cost functional as an ℓ^2 norm of the cost functionals in Equation (11) for each of the colour channels, resulting in

$$E = \sqrt{\sum_i E_i^2}, \tag{16}$$

where E_i is the total variation for each colour channel according to Equation (11).

Later approaches have been based on the structure tensor by Di Zenzo [33] and Bugun and Granlund [34], with components

$$s_{ij} = u^{\mu}_{,i} u^{\mu}_{,j}. \tag{17}$$

Sapiro and Ringach [35] proposed to use the eigenvalues

$$\lambda^{\pm} = \frac{1}{2}\left(s_{11} + s_{22} \pm \sqrt{(s_{11} - s_{22})^2 + 4s_{12}^2}\right) \tag{18}$$

and the corresponding eigenvectors $\theta_{\pm}$ of the structure tensor as a basis for constructing the diffusion equations. In terms of these eigenvalues, an alternative to the colour total variation by Blomgren and Chan can be obtained by the cost functional

$$E = \int_{\Omega} \sqrt{\lambda^+ + \lambda^-}\, d\Omega = \int_{\Omega} \sqrt{s}\, d\Omega. \tag{19}$$

For greyscale images, where $\lambda^+ = |\nabla u|^2$ and $\lambda^- = 0$, this reduces to total variation. For colour images, the corresponding Euler–Lagrange equations become

$$\frac{\partial u^{\mu}}{\partial t} = \partial_i \left(\frac{u^{\mu}_{,i}}{\sqrt{u^{\nu}_{,j} u^{\nu}_{,j}}} \right), \tag{20}$$

which again is on the form of Equation (7) with $D(s) = 1/\sqrt{s} = 1/\sqrt{\lambda^+ + \lambda^-} = 1/\sqrt{s_{11} + s_{22}} = 1/\sqrt{u^{\nu}_{,j} u^{\nu}_{,j}}$. Notice the coupling between the colour channels introduced by the sum in the denominator of Equation (20).

Tschumperlé and Deriche [4] extended this approach to *anisotropic* diffusion by introducing the general Lagrangian density $\psi(\lambda^+, \lambda^-)$ in the cost functional

$$E = \int_{\Omega} \psi(\lambda^+, \lambda^-)\, d\Omega. \tag{21}$$

The corresponding Euler–Lagrange equations are

$$\frac{\partial u^{\mu}}{\partial t} = \partial_k (D^{kl} u^{\mu}_{,l}), \tag{22}$$

where D^{kl} are the components of the diffusion tensor

$$D^{kl} = 2 \frac{\partial \psi}{\partial \lambda^{\pm}} \theta^k_{\pm} \theta^l_{\pm} \tag{23}$$

and $\theta_{\pm}$ are the orthonormal eigenvectors of the structure sensor s_{ij}, summing also over the eigenvalue and eigenvector indices for convenience (even though they, technically speaking, are not coordinates). It should be noted that this encompasses all previously presented diffusion methods as follows: $\psi(\lambda^+, \lambda^-) = \sqrt{\lambda^+ + \lambda^-}$ gives the solution of Sapiro and Ringach [35] for colour images and total variation in Rudin, Osher, and Fatemi [32] for greyscale images, $\psi(\lambda^+, \lambda^-) = -K^2 \exp(-s/K^2)$ and $\psi(\lambda^+, \lambda^-) = K^2 \ln(1 + s/K^2)$ gives the two equations of Perona and Malik [2], and $\psi(\lambda^+, \lambda^-) = s/2$ gives the classical linear diffusion of Koenderink [1]. In general, choosing $\psi(\lambda^+, \lambda^-) = \phi(\lambda^+ + \lambda^-) = \phi(s)$ leads to isotropic equations, since then $\partial\psi/\partial\lambda^+ = \partial\psi/\partial\lambda^- = \phi'(s)$ and the diffusion tensor in Equation (23) reduces to the scalar diffusion coefficient $D = 2\phi'(s)$. With $\psi(\lambda^+, \lambda^-) = \phi^+(\lambda^+) + \phi^-(\lambda^-)$, the diffusion in the mutually orthogonal directions of maximal and minimal change can be controlled independently, like used in Farup [11].

All of the above applies to Euclidean colour spaces. Renner [36] extended the approach of Tschumperlé and Deriche for the *isotropic* case[2] of $\psi(\lambda^+, \lambda^-) = \phi(s)$ to Riemannian geometries with *diagonal metric tensors*, and found that in that case, the diffusion equation can be written

$$\frac{\partial u^{\rho}}{\partial t} = \partial_k (D u^{\rho}_{,k}) + D \Gamma^{\rho}_{\mu\nu} u^{\mu}_{,k} u^{\nu}_{,k} \tag{24}$$

with $D = 2\phi'(s)$ as the diffusion coefficient. Here, $\Gamma^{\rho}_{\mu\nu}$ denotes the components of the Christoffel symbols of the Riemannian colour geometry (see, e.g., [22]).

Sochen et al. [37] constructed a geometrical framework for flow for images. Considering images as two-dimensional manifolds with coordinates X^i embedded in five-dimensional manifolds (two spatial and three colour dimensions), they showed how the Polyakov action [38] (using Sochen's original notation)

$$S[X^i, g_{\mu\nu}, h_{ij}] = \int D^m \sigma \sqrt{g} g^{\mu\nu} \partial_{\mu} x^i \partial_{\nu} X^j h_{ij}(\mathbf{X}), \tag{25}$$

where g is the image metric and h the metric of the embedding space, led to the Euler–Lagrange equations

$$\frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \partial_{\nu} X^i) + \Gamma^i_{jk} \partial_{\mu} X^j \partial_{\nu} X^k g^{\mu\nu} = 0. \tag{26}$$

However, the connection of this framework to the diffusion tensor of anisotropic diffusion, and thus the most general case of anisotropic diffusion in Riemannian geometries with non-diagonal metrics, has not yet been covered in the literature.

2.5 Data Attachment for Colour Images

Just as for grey-scale images, all the diffusion equations described above will produce a homogeneous flat image as $t \to \infty$ when solved with natural boundary conditions. Rudin, Osher, and Fatemi [32] used a data attachment term

[2] Also in the work of Renner [36], the method was termed anisotropic despite being isotropic, cf. the previously discussed confusion.

based on the ℓ^2 norm of the difference between the unknown image and the initial value,

$$E = \frac{\omega}{2} \int_\Omega (u^\mu - u_0^\mu)(u^\mu - u_0^\mu) d\Omega \tag{27}$$

leading to an extra term $\omega(u^\mu - u_0^\mu)$ in the PDE for Equation (22)

$$\frac{\partial u^\mu}{\partial t} = \partial_k (D^{kl} u^\mu_{,l}) - \omega(u^\mu - u_0^\mu). \tag{28}$$

Pang et al. [39] suggested using the ℓ^1 norm instead,

$$E = \omega \int_\Omega \sqrt{(u^\mu - u_0^\mu)(u^\mu - u_0^\mu)}\, d\Omega \tag{29}$$

leading to the extra term $\omega(u^\mu - u_0^\mu)/\sqrt{(u^\nu - u_0^\nu)(u^\nu - u_0^\nu)}$ in the Euler–Lagrange equations. E.g. Equation (22), this gives

$$\frac{\partial u^\mu}{\partial t} = \partial_k (D^{kl} u^\mu_{,l}) - \omega \frac{u^\mu - u_0^\mu}{\sqrt{(u^\nu - u_0^\nu)(u^\nu - u_0^\nu)}}. \tag{30}$$

3 Anisotropic Diffusion in Riemannian Colour Geometry

3.1 Basic Assumptions

To make the results as general as possible, let $\Omega \subset \mathbb{R}^N$ denote the N-dimensional (typically two) Euclidean image domain, $\mathcal{C}$ denote an M-dimensional (typically three) Riemannian colour manifold, and $u : \Omega \to \mathcal{C}$ denote an image with components u^μ. Let

$$s_{ij} = g_{\mu\nu} u^\mu_{,i} u^\nu_{,j} \tag{31}$$

with N eigenvalues λ^p and orthonormal eigenvectors θ_p defining the natural Riemannian generalisation of the structure tensor. The eigenvalue decomposition of s is given as[3]

$$s_{kl} = \lambda^p \theta_p^k \theta_p^l. \tag{32}$$

To avoid ambiguity, we assume the generic condition, which means distinct eigenvalues in all points. Thus, the eigenvectors θ_p are well defined up to sign. The eigenvector fields θ_p are differentiable with the generic condition. That is not the case in the non-generic case in general. In fact, the function $u(x, y) = xy$ provides a greyscale example where it is impossible to find continuous fields of eigenvectors for s. Because of the continuity of final Equation (45), this has no consequences for the validity of the result in the general case.

[3] We apply Einstein's summation convention for the index p referring to different corresponding eigenvalues and eigenvectors as well, although it is not a coordinate index.

3.2 General Formulation

Introduce the Lagrangian density $\psi(\lambda)$ as a function solely of the eigenvalues of the structure tensor. Then, the cost functional

$$E(u) = \int_\Omega \psi(\lambda)\, d\Omega \tag{33}$$

is minimised by the Euler–Lagrange equations

$$\partial_i \left(\frac{\partial \psi}{\partial u^\rho_{,i}} \right) - \frac{\partial \psi}{\partial u^\rho} = 0. \tag{34}$$

To derive the explicit form of the Euler–Lagrange Equation (34), we will need

$$\frac{\partial \psi}{\partial u^\rho} = \frac{\partial \psi}{\partial \lambda^p} \frac{\partial \lambda^p}{\partial s_{kl}} \frac{\partial s_{kl}}{\partial u^\rho} \tag{35}$$

$$\frac{\partial \psi}{\partial u^\rho_{,i}} = \frac{\partial \psi}{\partial \lambda^p} \frac{\partial \lambda^p}{\partial s_{kl}} \frac{\partial s_{kl}}{\partial u^\rho_{,i}}, \tag{36}$$

where the two first factors are common.

The first factor of the two expressions, $\partial\psi/\partial\lambda^p$, can be computed directly since the Lagrangian ψ is defined explicitly in terms of the eigenvalues of the structure tensor, and will thus depend on the design of the Lagrangian density.

The second factor of the two expressions can be computed implicitly following the method of Tschumperlé and Deriche [4] as follows

$$\begin{aligned} \delta_i^k \delta_j^l &= \frac{\partial s_{kl}}{\partial s_{ij}} \\ &= \frac{\partial \lambda^p}{\partial s_{ij}} \theta_p^k \theta_p^l + \lambda^p \frac{\partial \theta_p^k}{\partial s_{ij}} \theta_p^l + \lambda^p \theta_p^k \frac{\partial \theta_p^l}{\partial s_{ij}}. \end{aligned} \tag{37}$$

Multiplying with θ_m^k and θ_m^l, summing over the k and l indices, using that $\theta_m^k \theta_p^k = \delta_{pm}$, and $\theta_p^k (\partial \theta_p^k / \partial s_{ij}) = 0$, the latter due to the orthonormality of the θs, give

$$\frac{\partial \lambda^m}{\partial s_{ij}} = \theta_m^i \theta_m^j \tag{38}$$

(no sum).

The last factors of the two expressions, $\partial s_{kl}/\partial u^\rho$ and $\partial s_{kl}/\partial u^\rho_{,i}$ are, in general, different from the Euclidean case:

$$\frac{\partial s_{kl}}{\partial u^\rho} = \frac{\partial g_{\mu\nu}}{\partial u^\rho} u^\mu_{,k} u^\nu_{,l} \tag{39}$$

$$\frac{\partial s_{kl}}{\partial u^\rho_{,i}} = g_{\rho\mu}(\delta^k_i u^\mu_{,l} + \delta^l_i u^\mu_{,k}). \tag{40}$$

Inserting Equation (38) to (40) into Equation (35) and (36) gives

$$\begin{aligned}\frac{\partial \psi}{\partial u^\rho} &= \frac{\partial \psi}{\partial \lambda^p}\theta^k_p \theta^l_p \frac{\partial g_{\mu\nu}}{\partial u^\rho} u^\mu_{,k} u^\nu_{,l} \\ &= \frac{1}{2} D^{kl} \frac{\partial g_{\mu\nu}}{\partial u^\rho} u^\mu_{,k} u^\nu_{,l}\end{aligned} \tag{41}$$

$$\begin{aligned}\frac{\partial \psi}{\partial u^\rho_{,k}} &= 2\frac{\partial \psi}{\partial \lambda^p}\theta^k_p \theta^l_p g_{\rho\mu} u^\mu_{,l} \\ &= D^{kl} g_{\rho\mu} u^\mu_{,l},\end{aligned} \tag{42}$$

where the diffusion tensor

$$D^{kl} = 2\frac{\partial \psi}{\partial \lambda^p}\theta^k_p \theta^l_p \tag{43}$$

has been introduced. Inserted into the Euler–Lagrange Equation (34), this gives (showing some intermediate steps to illustrate the index gymnastics leading to the Christoffel symbols)

$$\begin{aligned}0 &= \partial_k\left(\frac{\partial \psi}{\partial u^\rho_{,k}}\right) - \frac{\partial \psi}{\partial u^\rho} \\ &= \partial_k\left(D^{kl} g_{\rho\mu} u^\mu_{,l}\right) - \frac{1}{2} D^{kl} \frac{\partial g_{\mu\nu}}{\partial u^\rho} u^\mu_{,k} u^\nu_{,l} \\ &= g_{\rho\mu}\partial_k\left(D^{kl} u^\mu_{,l}\right) + D^{kl}\frac{\partial g_{\rho\mu}}{\partial u^\nu} u^\mu_{,l} u^\nu_{,k} \\ &\quad - \frac{1}{2} D^{kl} \frac{\partial g_{\mu\nu}}{\partial u^\rho} u^\mu_{,k} u^\nu_{,l} \\ &= g_{\rho\mu}\partial_k\left(D^{kl} u^\mu_{,l}\right) \\ &\quad + D^{kl}\frac{1}{2}\left(\frac{\partial g_{\rho\mu}}{\partial u^\nu} + \frac{\partial g_{\rho\nu}}{\partial u^\mu} - \frac{\partial g_{\mu\nu}}{\partial u^\rho}\right) u^\mu_{,k} u^\nu_{,l} \\ &= g_{\rho\mu}\partial_k(D^{kl} u^\mu_{,l}) + D^{kl}\Gamma_{\rho\mu\nu} u^\mu_{,k} u^\nu_{,l},\end{aligned} \tag{44}$$

where $\Gamma_{\rho\mu\nu} = (\partial g_{\rho\mu}/\partial u^\nu + \partial g_{\rho\nu}/\partial u^\mu - \partial g_{\mu\nu}/\partial u^\rho)/2$ denote the Christoffel symbols of the first kind and the symmetries of both D^{kl} and $g_{\mu\nu}$ have been used between the third and fourth lines.

Using the metric tensor to convert to contravariant vector components ('raise the index') and solving by gradient descent gives the more convenient form for our main result,

$$\frac{\partial u^\rho}{\partial t} = \partial_k(D^{kl} u^\rho_{,l}) + D^{kl}\Gamma^\rho_{\mu\nu} u^\mu_{,k} u^\nu_{,l}, \tag{45}$$

where $\Gamma^\rho_{\mu\nu} = g^{\rho\kappa}\Gamma_{\kappa\mu\nu}$ denote the Christoffel symbols of the second kind. In general, this constitutes a set of coupled nonlinear partial differential equations.

It should be noted that the result is in complete agreement with the results of Tschumperlé and Deriche [4] for the Euclidean case where the Christoffel symbols vanish, and with the results of Renner [36] for the isotropic case—also in the case where the metric tensor is not diagonal, as required in the proof of Renner—using $\psi(\lambda) = \phi(\lambda^+ + \lambda^-) = \phi(s)$.

4 Discussion

4.1 Proof of Coordinate Independence

It is not obvious that Equation (45) is a tensor equation, i.e. that it is independent of the choice of coordinates. To prove that it indeed is, we will check how the Euler–Lagrange equation transforms. Given coordinate change $v^{\bar\rho} = v^{\bar\rho}(u^1, u^2, \ldots, u^M)$. The chain rule gives

$$v^{\bar\mu}_{,i} = \frac{\partial v^{\bar\mu}}{\partial u^\eta} u^\eta_{,i}. \tag{46}$$

The chain rule applied to the second term of the Euler–Lagrange equation gives

$$\begin{aligned}\frac{\partial \psi}{\partial u^\rho} &= \frac{\partial \psi}{\partial v^{\bar\mu}}\frac{\partial v^{\bar\mu}}{\partial u^\rho} + \frac{\partial \psi}{\partial v^{\bar\mu}_{,i}}\frac{\partial v^{\bar\mu}_{,i}}{\partial u^\rho} \\ &= \frac{\partial \psi}{\partial v^{\bar\mu}}\frac{\partial v^{\bar\mu}}{\partial u^\rho} + \frac{\partial \psi}{\partial v^{\bar\mu}_{,i}}\frac{\partial^2 v^{\bar\mu}}{\partial u^\rho \partial u^\eta} u^\eta_{,i}.\end{aligned} \tag{47}$$

The chain rule applied to the first term of the Euler–Lagrange equation gives

$$\frac{\partial \psi}{\partial u^\rho_{,i}} = \frac{\partial \psi}{\partial v^{\bar\mu}_{,j}}\frac{\partial v^{\bar\mu}_{,j}}{\partial u^\rho_{,i}} = \frac{\partial \psi}{\partial v^{\bar\mu}_{,i}}\frac{\partial v^{\bar\mu}}{\partial u^\rho} \tag{48}$$

and

$$\begin{aligned}\partial_i\left(\frac{\partial \psi}{\partial u^\rho_{,i}}\right) &= \partial_i\left(\frac{\partial \psi}{\partial v^{\bar\mu}_{,i}}\right)\frac{\partial v^{\bar\mu}}{\partial u^\rho} + \frac{\partial \psi}{\partial v^{\bar\mu}_{,i}}\partial_i\left(\frac{\partial v^{\bar\mu}}{\partial u^\rho}\right) \\ &= \partial_i\left(\frac{\partial \psi}{\partial v^{\bar\mu}_{,i}}\right)\frac{\partial v^{\bar\mu}}{\partial u^\rho} + \frac{\partial \psi}{\partial v^{\bar\mu}_{,i}}\frac{\partial^2 v^{\bar\mu}}{\partial u^\eta \partial u^\rho} u^\eta_{,i}.\end{aligned} \tag{49}$$

By combining these formulas, we get

$$\partial_i\left(\frac{\partial \psi}{\partial u^\rho_{,i}}\right) - \frac{\partial \psi}{\partial u^\rho} = \frac{\partial v^{\bar\mu}}{\partial u^\rho}\left[\partial_i\left(\frac{\partial \psi}{\partial v^{\bar\mu}_{,i}}\right) - \frac{\partial \psi}{\partial v^{\bar\mu}}\right]. \tag{50}$$

The metric tensor $g_{\rho\mu}$ transforms as

$$g_{\rho\nu} = \frac{\partial v^{\bar{\eta}}}{\partial u^{\rho}} \frac{\partial v^{\bar{\mu}}}{\partial u^{\nu}} g_{\bar{\eta}\bar{\mu}} \tag{51}$$

and the dual tensor $g^{\rho\mu}$ transforms as

$$g^{\rho\nu} = \frac{\partial u^{\rho}}{\partial v^{\bar{\eta}}} \frac{\partial u^{\nu}}{\partial v^{\bar{\mu}}} g^{\bar{\eta}\bar{\mu}}. \tag{52}$$

Therefore,

$$\begin{aligned} g^{\rho\nu} &\left[\partial_i \left(\frac{\partial \psi}{\partial u^{\nu}_{,i}} \right) - \frac{\partial \psi}{\partial u^{\nu}} \right] \\ &= g^{\rho\nu} \frac{\partial v^{\bar{\sigma}}}{\partial u^{\nu}} \left[\partial_i \left(\frac{\partial \psi}{\partial v^{\bar{\sigma}}_{,i}} \right) - \frac{\partial \psi}{\partial v^{\bar{\sigma}}} \right] \\ = \frac{\partial u^{\rho}}{\partial v^{\bar{\eta}}} \frac{\partial u^{\nu}}{\partial v^{\bar{\mu}}} g^{\bar{\eta}\bar{\mu}} \frac{\partial v^{\bar{\sigma}}}{\partial u^{\nu}} &\left[\partial_i \left(\frac{\partial \psi}{\partial v^{\bar{\sigma}}_{,i}} \right) - \frac{\partial \psi}{\partial v^{\bar{\sigma}}} \right] \\ = \frac{\partial u^{\rho}}{\partial v^{\bar{\eta}}} &\left(g^{\bar{\eta}\bar{\mu}} \left[\partial_i \left(\frac{\partial \psi}{\partial v^{\bar{\mu}}_{,i}} \right) - \frac{\partial \psi}{\partial v^{\bar{\mu}}} \right] \right). \end{aligned} \tag{53}$$

The last equality comes from $\delta^{\bar{\sigma}}_{\bar{\mu}} = \partial v^{\bar{\sigma}} / \partial v^{\bar{\mu}} = (\partial v^{\bar{\sigma}} / \partial u^{\nu}) (\partial u^{\nu} / \partial v^{\bar{\mu}})$ On the other hand, du^{ρ}/dt transforms in the same manner,

$$\frac{du^{\rho}}{dt} = \frac{\partial u^{\rho}}{\partial v^{\bar{\eta}}} \frac{dv^{\bar{\eta}}}{dt}, \tag{54}$$

showing that Equation (45) is indeed a tensor equation.

4.2 Connection with Sochen's Framework

Sochen et. al. [37] allow the image coordinates to change. Sochen's coordinates σ^1 and σ^2 are not image points but coordinates of an embedded manifold $\mathbf{X} : \Sigma \to M$ representing the image, where Σ is the image plane and M is a product of the image plane and a colour space. Also, Sochen's metric on Σ is not the metric inherited by the embedding into M. Sochen uses lowercase Roman letters for the coordinates of M and Greek letters for the coordinates in Σ. We choose to use marked Greek letters for coordinates on M and marked Roman lowercase letters for Σ to avoid confusion with our notion. In our setting, M is the direct product of the colour space and the image plane. In Sochen's framework, the image points are functions of time and these coordinates, $x^i(\sigma^1, \sigma^2, t) = X^{\rho'}(\sigma^1, \sigma^2, t)$, $i = \rho' = 1, 2$. Sochen's formulas (8) and (9) for $i = 1, 2$ therefore give

$$\frac{\partial x^i}{\partial t} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \sigma^{i'}} \left(\sqrt{g} g^{i'j'} \frac{\partial x^i}{\partial \sigma^{j'}} \right). \tag{55}$$

Our colour coordinates u^1, u^2 and u^3 are functions of x^1, x^2 and t. In Sochen's framework, that corresponds to

$$X^{\mu+2}(\sigma^1, \sigma^2, t) = u^{\mu}(x^1(\sigma^1, \sigma^2, t), x^2(\sigma^1, \sigma^2, t), t).$$

For $\rho' = 2 + \rho$, $\rho = 1, 2, 3$, Sochen's formula is

$$\begin{aligned} \frac{\partial X^{\rho'}}{\partial t} =& \frac{1}{\sqrt{g}} \partial_{i'} (\sqrt{g} g^{i'j'} \partial_{j'} X^{\rho'}) \\ &+ \Gamma^{\rho'}{}_{\mu'\nu'} \partial_{i'} X^{\mu'} \partial_{j'} X^{\nu'} g^{i'j'}. \end{aligned} \tag{56}$$

Combining the chain rule $\frac{\partial X^{\rho+2}}{\partial t} = \frac{\partial u^{\rho}}{\partial x^i} \frac{\partial x^i}{\partial t} + \frac{\partial u^{\rho}}{\partial t}$ with Equation (55) and Equation (56) gives

$$\begin{aligned} \frac{\partial u^{\rho}}{\partial t} =& -\frac{\partial u^{\rho}}{\partial x^i} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \sigma^{i'}} \left(\sqrt{g} g^{i'j'} \frac{\partial x^i}{\partial \sigma^{j'}} \right) \\ &+ \frac{1}{\sqrt{g}} \frac{\partial}{\partial \sigma^{i'}} \left(\sqrt{g} g^{i'j'} \frac{\partial x^i}{\partial \sigma^{j'}} \frac{\partial u^{\rho}}{\partial x^i} \right) \\ &+ \Gamma^{\rho}_{\kappa\gamma} \left(\frac{\partial x^i}{\partial \sigma^{i'}} g^{i'j'} \frac{\partial x^j}{\partial \sigma^{j'}} \right) \frac{\partial u^{\kappa}}{\partial x^i} \frac{\partial u^{\gamma}}{\partial x^j}, \end{aligned} \tag{57}$$

where $\nu' = 2+\kappa$ and $\mu' = 2+\gamma$. The product rule simplifies the first two lines of the equation.

$$\begin{aligned} \frac{\partial u^{\rho}}{\partial t} =& g^{i'j'} \frac{\partial x^i}{\partial \sigma^{j'}} \frac{\partial}{\partial \sigma^{i'}} \left(\frac{\partial u^{\rho}}{\partial x^i} \right) \\ &+ \Gamma^{\rho}_{\kappa\gamma} \left(\frac{\partial x^i}{\partial \sigma^{i'}} g^{i'j'} \frac{\partial x^j}{\partial \sigma^{j'}} \right) \frac{\partial u^{\kappa}}{\partial x^i} \frac{\partial u^{\gamma}}{\partial x^j} \end{aligned} \tag{58}$$

Finally, applying the chain rule $\frac{\partial}{\partial \sigma^{i'}} = \frac{\partial x^i}{\partial \sigma^{i'}} \frac{\partial}{\partial x^i}$, we obtain

$$\begin{aligned} \frac{\partial u^{\rho}}{\partial t} =& g^{i'j'} \left(\frac{\partial x^i}{\partial \sigma^{i'}} g^{i'j'} \frac{\partial x^j}{\partial \sigma^{j'}} \right) \frac{\partial^2 u^{\rho}}{\partial x^i \partial x^j} \\ &+ \Gamma^{\rho}_{\kappa\gamma} \left(\frac{\partial x^i}{\partial \sigma^{i'}} g^{i'j'} \frac{\partial x^j}{\partial \sigma^{j'}} \right) \frac{\partial u^{\kappa}}{\partial x^i} \frac{\partial u^{\gamma}}{\partial x^j}. \end{aligned} \tag{59}$$

With the choice $D^{ij} = g^{ij} = \frac{\partial x^i}{\partial \sigma^{i'}} g^{i'j'} \frac{\partial x^j}{\partial \sigma^{j'}}$, our method fits in the Sochen's framework. Notice that with this choice, the inverse of D^{ij} is not the inherited metric of the embedding $\mathbf{X}$.

4.3 Data Attachment

For the Euclidean case, data attachment is introduced as the ℓ^2 or ℓ^1 norm of the colour differences, cf. Equation (27) and (29). The norm is used to measure the difference of two colours at, in general, two different positions of the colour manifold. In Riemannian geometry, a closed-form expression for distance is, in general, only available infinitesimally through the metric, $dl^2 = g_{\mu\nu} du^\mu du^\nu$. Finite distances are measured along the geodesic curve $\gamma(t)$,

$$d(u, u_0) = \int_{u_0}^{u} \sqrt{g_{\mu\nu} d\gamma^\mu d\gamma^\nu}, \quad (60)$$

which in turn is found by solving the geodesic equation for the geometry in question,

$$\frac{D^2\gamma^\rho}{dt^2} + \Gamma^\rho_{\mu\nu} \frac{d\gamma^\mu}{dt} \frac{d\gamma^\nu}{dt} = 0. \quad (61)$$

In practice, these equations must be solved numerically, see, e.g., Chevallier and Farup [40], adding significantly to the complexity of the numerical solution. Fortunately, when applying a pure anisotropic diffusion algorithm to regularise an image, it is common practice not to consider any attachment term to the data but only apply a finite number of iterations until the level of regularisation is sufficient.

4.4 Special Case: Hyperbolic Geometry

For most actual Riemannian geometries, the full explicit expression of Equation (45) will be very complex to implement due to the complexity of the Christoffel symbols $\Gamma^\rho_{\mu\nu}$. However, in the special case of hyperbolic colour geometry, an explicit expression of the equations is within reach. Consider the Poincaré half-plane model $\mathbb{R} \times \mathbb{H}$ of hyperbolic geometry with the coordinate substitution $u^0 \mapsto \ln u^0$. The nonzero components of the metric tensor (cf. Equation (5)) and its derivatives are

$$\begin{aligned} g_{00} &= 1 \\ g_{11} = g_{22} &= \frac{\alpha}{(u^2)^2} \\ g_{11,2} = g_{22,2} &= -2\alpha/(u^2)^3, \end{aligned} \quad (62)$$

giving the nonzero Christoffel symbols

$$\begin{aligned} \Gamma^2_{11} &= \frac{1}{u^2} \\ \Gamma^1_{21} = \Gamma^1_{12} = \Gamma^2_{22} &= -\frac{1}{u^2}, \end{aligned} \quad (63)$$

which in turn give the Euler–Lagrange equations,

$$\begin{aligned} \frac{\partial u^0}{\partial t} &= \partial_k (D^{kl} u^0_{,l}) \\ \frac{\partial u^1}{\partial t} &= \partial_k (D^{kl} u^1_{,l}) - D^{kl} \frac{2u^1_{,k} u^2_{,l}}{u^2} \\ \frac{\partial u^2}{\partial t} &= \partial_k (D^{kl} u^2_{,l}) + D^{kl} \frac{u^1_{,k} u^1_{,l} - u^2_{,k} u^2_{,l}}{u^2}. \end{aligned} \quad (64)$$

It should be noted that even for the presumably simple case of isotropic non-local diffusion with $\psi(\lambda^+, \lambda^-) = s/2$ giving $D^{kl} = \delta^{kl}$, the resulting equations are coupled and nonlinear for the two chromatic colour coordinates, in contrast with the Euclidean case, where everything is linear and decoupled.

For data attachment in this special case of the $\mathbb{R} \times \mathbb{H}$ hyperbolic geometry, however, the following closed-form solution of the distance, Equation (60), between two points is known,

$$\begin{aligned} d(u, u_0)^2 &= (u^0 - u^0_0)^2 + \\ &\alpha^2 \operatorname{arcosh}^2 \left(1 + \frac{(u^1 - u^1_0)^2 + (u^2 - u^2_0)^2}{2u^2 u^2_0} \right). \end{aligned} \quad (65)$$

Basing the data attachment term on this (in analogy to using the ℓ^2 norm in Euclidean geometry) by adding

$$E = \frac{\omega}{2} \int_\Omega d(u, u_0)^2 \, d\Omega \quad (66)$$

to the cost functional gives the following extensions to the general Euler–Lagrange Equation (64)

$$\begin{aligned} \frac{\partial u^0}{\partial t} &= \partial_k (D^{kl} u^0_{,l}) + \omega(u^0 - u^0_0) \\ \frac{\partial u^1}{\partial t} &= \partial_k (D^{kl} u^1_{,l}) - D^{kl} \frac{2u^1_{,k} u^2_{,l}}{u^2} \\ &\quad - \omega\alpha^2 \operatorname{arcosh} \left(1 + \frac{(u^1 - u^1_0)^2 + (u^2 - u^2_0)^2}{2u^2 u^2_0} \right) \\ &\quad \times \left(\left(1 + \frac{(u^1 - u^1_0)^2 + (u^2 - u^2_0)^2}{2u^2 u^2_0} \right)^2 - 1 \right)^{-1/2} \\ &\quad \times \frac{u^1 - u^1_0}{u^2 u^2_0} \\ \frac{\partial u^2}{\partial t} &= \partial_k (D^{kl} u^2_{,l}) + D^{kl} \frac{u^1_{,k} u^1_{,l} - u^2_{,k} u^2_{,l}}{u^2} \\ &\quad - \omega\alpha^2 \operatorname{arcosh} \left(1 + \frac{(u^1 - u^1_0)^2 + (u^2 - u^2_0)^2}{2u^2 u^2_0} \right) \\ &\quad \times \left(\left(1 + \frac{(u^1 - u^1_0)^2 + (u^2 - u^2_0)^2}{2u^2 u^2_0} \right)^2 - 1 \right)^{-1/2} \\ &\quad \times \frac{(u^2)^2 - (u^2_0)^2 - (u^1 - u^1_0)^2}{2(u^2)^2 u^2_0}. \end{aligned} \quad (67)$$

Even more extensive expressions will result from the analogue of the ℓ^1 case with the cost functional $E = \omega \int_\Omega d(u, u_0) \, d\Omega$.

5 Conclusion and Future Work

The equations for anisotropic diffusion in Riemannian colour geometry are derived both for the general case and for the special case of hyperbolic geometry. This includes the possible treatment of a data attachment term. The non-trivial connection with Sochen's framework [37] is elaborated. Several other well-known solutions appear as special cases.

Future work should look into applying the technique to various real-life colour imaging applications such as image denoising, image inpainting, and colour filter array demosaicing using the most up-to-date colour geometries. This includes the development of computationally efficient algorithms for solving Equation (45) numerically for real cases, which is not a straightforward and trivial task. An extension to gradient domain image processing, requiring a method to deal with gradients defined in different tangent spaces of the colour manifold, remains an open research question.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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